

dS/CFT Correspondence from a Holographic Description of Massless Scalar Fields in Minkowski Space-Time

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Abstract

We solve Klein-Gordon equation for massless scalars on $d+1$ dimensional Minkowski (Euclidean) space in terms of the Cauchy data on the hypersurface $t=0$. By inserting the solution into the action of massless scalars in Minkowski (Euclidean) space we obtain the action of dual theory on the boundary $t=0$ which is exactly the holographic dual of conformally coupled scalars on $d+1$ dimensional (Euclidean anti) de Sitter space obtained in (A)dS/CFT correspondence. The observed equivalence of dual theories is explained using the one-to-one map between conformally coupled scalar fields on Minkowski (Euclidean) space and (Euclidean anti) de Sitter space which is an isomorphism between the hypersurface $t=0$ of Minkowski (Euclidean) space and the boundary of (A)dS space.

1 Introduction

The holographic principle is in general a duality between the physics of a generic space-time and a theory on the boundary (the holographic screen) [1]. Quantitative examples of the holographic principle are AdS/CFT [2, 3] and dS/CFT [4, 5] correspondence. Recently de Boer and Solodukhin proposed a holographic description of Minkowski space in terms of a dual CFT defined on the boundary of the light cone [6].

In this paper we give a holographic description of massless scalar field on $d+1$ dimensional Minkowski (Euclidean) space in terms of a dual theory on the hypersurface $t = 0$.

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The dual action is obtained by inserting the solution of Klein-Gordon equation into the action of massless scalars on Minkowski (Euclidean) space. As is shown, the action on the boundary is equivalent to the action of the theory on the boundary of (Euclidean anti) de Sitter space dual to conformally coupled scalars obtained in (A)dS/CFT correspondence. The equivalence of dual theories can be explained by using the injective map between massless scalars on Minkowski (Euclidean) space and dS (Euclidean AdS) space with the same dimensionality which induces an isomorphism between the hypersurface $t=0$ of Minkowski (Euclidean) space and the boundary of (Euclidean anti) de Sitter space.

The organization of the paper is as follows. In section 2 we study the correspondence between massless scalars in $d+1$ dimensional Euclidean space R^{d+1} and scalars with mass $m^2 = \frac{1-d^2}{4}$ (the conformally coupled scalars) in Euclidean AdS_{d+1} . Section 3 is devoted to scalars on Minkowski space and dS/CFT correspondence. The notation used in sections 2 and 3 are similar to the notation of references [3] and [4] respectively. In section 3, we describe the \mathcal{O}^- region of dS_{d+1} by metric, $ds^2 = t^{-2}(-dt^2 + dx_i^2)$ which is related to the metric $ds^2 = -du^2 + e^{-2u}dx_i^2$ used in [4] by the identity $t = e^u$. Most significant properties of dS space related to dS/CFT correspondence are studied in [7]. Various aspects of AdS/CFT and dS/CFT duality are recently reviewed in [8]. In section 4, we summarize our results and address some of applications of the introduced method. The scalar field theory in curved spacetime is briefly reviewed in the appendix.

2 Massless Scalars on Euclidean Space

The equation of motion for massive scalar fields on Euclidean AdS_{d+1} with metric,

$$ds^2 = \frac{1}{t^2} \left(dt^2 + \sum_{i=1}^d dx_i^2 \right), \quad (1)$$

is

$$\left(t^2 \partial_t^2 + (1-d)t \partial_t + t^2 \nabla^2 - m^2 \right) \Phi = 0, \quad (2)$$

where $\partial_t = \frac{\partial}{\partial t}$ and

$$\nabla^2 = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}. \quad (3)$$

See the appendix for a brief review of scalar field theory in curved spacetimes. One can easily show that if ϕ is a massless scalar field in $d+1$ dimensional Euclidean space R^{d+1} with metric $ds^2 = dt^2 + dx_i^2$, satisfying the equation

$$(\partial_t^2 + \nabla^2) \phi = 0, \quad (4)$$

then

$$\Phi = t^{\frac{d-1}{2}} \phi, \quad (5)$$

is a solution of Eq.(2) with mass $m^2 = \frac{1-d^2}{4}$. Since $\frac{-d^2}{4} < m^2 < 0$, this solution is stable in AdS_{d+1} . From AdS/CFT correspondence [3] it is known that the dual theory on the boundary $t = 0$ is a conformal theory with the following action,

$$I[\phi] = \int d^d y d^d z \frac{\phi_0(\vec{y}) \phi_0(\vec{z})}{|\vec{y} - \vec{z}|^{d+\lambda_+}}, \quad (6)$$

where λ_+ is the larger root of the equation $\lambda(\lambda + d) = m^2$. Here $\lambda_+ = \frac{(1-d)}{2}$ as far as $m^2 = \frac{(1-d^2)}{4}$. ϕ_0 is a function on the boundary such that $\Phi(\vec{x}, t) \sim t^{-\lambda_+} \phi_0$ as $t \rightarrow 0$. From the map (5), one can interpret $\phi_0(\vec{x})$ in Euclidean space R^{d+1} as the initial data on the hypersurface $t = 0$. Therefore one expects that the action (6) can be obtained from the action of scalar fields in $d + 1$ dimensional Euclidean space,

$$I[\phi] = \frac{1}{2} \int dt d^d x \left((\partial_t \phi)^2 + (\nabla \phi)^2 \right), \quad (7)$$

if one solves equation (4) in terms of the initial data $\phi_0(\vec{x})$ given on the hypersurface $t = 0$. Proof is as follows:

The most general solution of the equation of motion that vanishes as t tends to infinity is

$$\phi(\vec{x}, t) = \int d^d k \tilde{\phi}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} e^{-\omega t}, \quad (8)$$

where $\omega = |\vec{k}|$ and

$$\tilde{\phi}(\vec{k}) = \int d^d x \phi_0(\vec{x}) e^{-i\vec{k} \cdot \vec{x}}. \quad (9)$$

Inserting (9) into (8) one obtains

$$\phi(\vec{x}, t) = \int d^d y \mathcal{G}(\vec{x}, t; \vec{y}) \phi_0(\vec{y}), \quad (10)$$

where

$$\mathcal{G}(\vec{x}, t; \vec{y}) = \int d^d k e^{-\omega t} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}. \quad (11)$$

$\mathcal{G}(\vec{x}, t; \vec{y})$ is the solution of wave equation i.e. $\square \mathcal{G} = 0$, with the initial condition $\mathcal{G}(\vec{x}, 0; \vec{y}) = \delta^d(\vec{x} - \vec{y})$. To obtain the action of the corresponding theory on the boundary $t = 0$ one should insert (10) into (7). But it is more suitable to rewrite the action (7) in the form,

$$I[\phi] = -\frac{1}{2} \int d^{d+1} x \phi \square \phi - \frac{1}{2} \int d^d x \phi_0(\vec{x}) \partial_t \phi_0(\vec{x}), \quad (12)$$

which is obtained by an integration by part and under the assumption that $\phi(x)$ vanishes as t tends to infinity and also at spatial infinity. Inserting (10) into (12), the first term vanishes and from the second term one obtains:

$$I[\phi] = \frac{1}{2} \int d^d y d^d z \phi_0(\vec{y}) \phi_0(\vec{z}) F(\vec{y} - \vec{z}), \quad (13)$$

in which

$$F(\vec{x}) = \int d^d k \, \omega e^{i\vec{k} \cdot \vec{x}}. \quad (14)$$

As can be verified from the rotational invariance ($\vec{x} \rightarrow \mathbf{R}\vec{x}$, $\mathbf{R} \in SO(d)$), $F(\vec{x})$ depends only on the norm of \vec{x} . By scaling \vec{x} by a factor $\lambda > 0$ one can also show that $F(\lambda\vec{x}) = \lambda^{-(d+1)}F(\vec{x})$ and consequently,

$$\vec{x} \cdot \nabla F(\vec{x}) = \lim_{\lambda \rightarrow 1} \frac{F(\lambda|\vec{x}|) - F(|\vec{x}|)}{\lambda - 1} = -(d+1)F(|\vec{x}|). \quad (15)$$

Therefore, $F(\vec{x}) = \text{Const.} |\vec{x}|^{-(d+1)}$ and

$$I[\phi] = \text{Const.} \int d^d y d^d z \frac{\phi_0(\vec{y})\phi_0(\vec{z})}{|\vec{y} - \vec{z}|^{d+1}}, \quad (16)$$

which is equal to (6) derived in AdS/CFT correspondence.

One should note that in addition to the general solution (8), Eq.(4) has one further solution $\phi(\vec{x}, t) = \alpha t$ where α is some constant that can not be determined from the initial data $\phi_0(\vec{x})$. Although this solution is meaningless in Euclidean space, but α is equal to the value of $\Phi(\vec{x}, t)$ at $t \rightarrow \infty$ which is an additional point on the boundary of AdS.

One can interpret the above result as a holographic principle for Euclidean space. But our analysis considers only massless scalars. As can be easily verified, it is not possible to generalize the map (5) to include massive scalar fields in R^{d+1} . A reason for this is the fact that the equation of motion of scalar fields $(\square + m^2)\phi = 0$ is not conformally covariant unless $m = 0$.

3 Massless Scalars on Minkowski Space-time

Similar to section 2, one can show that massless scalar fields in $d+1$ dimensional Minkowski space-time M_{d+1} can be mapped by (5) to scalars with mass $m^2 = \frac{d^2-1}{4}$ on dS_{d+1} . To verify this claim one can use the following metrics for dS_{d+1} and M_{d+1} respectively:

$$ds_{dS}^2 = \frac{1}{t^2} \left(-dt^2 + \sum_{i=1}^d dx_i^2 \right), \quad (17)$$

$$ds_M^2 = \left(-dt^2 + \sum_{i=1}^d dx_i^2 \right) \quad (18)$$

The metric (17) covers only half of dS space. This region called \mathcal{O}^- is the region observed by an observer on the south pole \mathcal{I}^- but is behind the horizon of the observer on the north pole \mathcal{I}^+ . By construction $t > 0$. Following the Strominger proposal, dual operators living on the boundary \mathcal{I}^- obey,

$$\langle \mathcal{O}_\phi(z, \bar{z}), \mathcal{O}_\phi(v, \bar{v}) \rangle = \frac{\text{const.}}{|z - v|^{2h_+}}, \quad (19)$$

where

$$h_{\pm} = \frac{1}{2} \left(d \pm \sqrt{d^2 - 4m^2} \right). \quad (20)$$

Again, the existence of the map (5) suggests that Eq.(19) can be obtained by solving the equations of motion of massless scalar fields on M_{d+1} in terms of the initial data at $t = 0$. If yes then the final result can be interpreted as a holographic description of massless scalars in Minkowski space time. Such a description can be made covariant by considering a covariant boundary [9] instead of the hypersurface $t = 0$, which here corresponds to the \mathcal{I}^- by (5).

General arguments [4] show that a massive scalar field behaves as $t^{h_{\pm}}\phi_{\pm}$ near \mathcal{I}^- and the dual theory on the boundary \mathcal{I}^- (the planar past region) is described by the action

$$I[\phi] = \int_{\mathcal{I}^-} d^d y d^d(z) \left(\frac{\phi_-(\vec{y})\phi_-(\vec{z})}{|\vec{y} - \vec{z}|^{2h_+}} + \frac{\phi_+(\vec{y})\phi_+(\vec{z})}{|\vec{y} - \vec{z}|^{2h_-}} \right). \quad (21)$$

Since $h_- = \frac{d-1}{2}$ for $m^2 = \frac{d^2-1}{4}$, using Eq.(5) one verifies that, $\phi_-(\vec{x}) = \phi(\vec{x}, t)|_{t=0}$. As will be exactly shown, $\phi_+(\vec{x}) = i\partial_t\phi(\vec{x}, t)|_{t=0}$. Two evidences for this claim are:

1. Since $h_+ = h_- + 1$ for $m^2 = \frac{d^2-1}{4}$, using Eq.(5) one can verify that $\partial_t\phi$ mapped to dS_{d+1} behaves as t^{h_+} near \mathcal{I}^- as demanded.
2. A general solution of equation of motion for massless scalars in M_{d+1} , contains oscillating terms with both positive and negative frequencies. Therefore $\phi(\vec{x}, t)$ can only be given in terms of both $\phi_0(\vec{x}, 0)$ and $\partial_t\phi_0(\vec{x}, 0)$.

In other words the main purpose of this section is to derive the action (21) of the dual theory by inserting the solution of Klein-Gordon equation for massless scalar fields in M_{d+1} ,

$$\left(\partial_t^2 - \nabla^2 \right) \phi = 0, \quad (22)$$

satisfying the initial conditions

$$\phi(\vec{x}, 0) = \phi_-(\vec{x}), \quad \partial_t\phi(\vec{x}, 0) = i\phi_+(\vec{x}), \quad (23)$$

into the action of massless scalar fields in M_{d+1} ,

$$I[\phi] = \frac{1}{2} \int dt d^d x \left((\partial_t\phi)^2 - (\nabla\phi)^2 \right). \quad (24)$$

The most general solution of Eq.(22) satisfying the initial conditions (23) is

$$\begin{aligned} \phi(\vec{x}, t) &= \frac{1}{2} \int d^d y d^d k \left(\phi_-(\vec{y}) - \frac{\phi_+(\vec{y})}{\omega} \right) e^{i\vec{k} \cdot (\vec{x} - \vec{y})} e^{-i\omega t} \\ &+ \frac{1}{2} \int d^d y d^d k \left(\phi_-(\vec{y}) + \frac{\phi_+(\vec{y})}{\omega} \right) e^{i\vec{k} \cdot (\vec{x} - \vec{y})} e^{i\omega t}, \end{aligned} \quad (25)$$

where $\omega = |\vec{k}|$. Inserting this solution into (24), one obtains that up to some constant coefficient,

$$I[\phi] = \frac{1}{2} \int d^d y d^d z \text{Re}[G(\vec{y}, \vec{z})] \quad (26)$$

where

$$G(\vec{y}, \vec{z}) = - \int d^d k (\omega \phi_-(\vec{y}) + \phi_+(\vec{y})) (\omega \phi_-(\vec{z}) + \phi_+(\vec{z})) e^{i\vec{k} \cdot (\vec{z} - \vec{y})} f(\omega) \quad (27)$$

in which,

$$\begin{aligned} f(\omega) &= \lim_{\alpha \rightarrow +0} \int_0^\infty dt e^{-(\alpha + 2i\omega)t} \\ &= \lim_{\alpha \rightarrow +0} \frac{1}{\alpha + 2i\omega} \end{aligned} \quad (28)$$

To obtain $\text{Re}[G(\vec{y}, \vec{z})]$, one should note that $f(\omega) = f^*(\omega)$. To prove this equality one can show that for any analytic function $g(\omega)$

$$\begin{aligned} \int_{-\infty}^\infty d\omega g(\omega) f(\omega) &= \int_{-\infty}^\infty d\omega g(\omega) f^*(\omega) \\ &= \lim_{\alpha \rightarrow +0} (-2\pi) g(\alpha). \end{aligned} \quad (29)$$

Consequently, up to some constant coefficients,

$$\begin{aligned} \text{Re}[G(\vec{y}, \vec{z})] &= \int d^d k \frac{\omega^2 \phi_0(\vec{y}) \phi_-(\vec{z}) + \phi_+(\vec{y}) \phi_+(\vec{z})}{\omega} e^{i\vec{k} \cdot (\vec{z} - \vec{y})} \\ &= \text{const.} \left(\frac{\phi_-(\vec{y}) \phi_-(\vec{z})}{|\vec{z} - \vec{y}|^{d+1}} + \frac{\phi_+(\vec{y}) \phi_+(\vec{z})}{|\vec{z} - \vec{y}|^{d-1}} \right). \end{aligned} \quad (30)$$

4 Conclusion

Considering the hypersurface $t=0$ as the boundary (the holographic screen) of Minkowski (Euclidean) space, we obtained the action of dual theory on the boundary by solving the Klein-Gordon equation of motion in terms of the Cauchy data and inserting the solution into the action of massless scalar fields in the bulk. Since massless scalars on Minkowski (Euclidean) space are in one-to-one correspondence to conformally coupled scalars on (Euclidean anti) de Sitter space with the same dimension one expects that (as is verified) the dual action is equal to the dual theory on the boundary of (A)dS space obtained in (A)dS/CFT correspondence.

Using this method one can study (A)dS/CFT duality in the case of self interacting conformally coupled scalar field theories, i.e $D=3$ ϕ^6 -model, $D=4$ ϕ^4 -model and $D=6$ ϕ^3 , by solving the corresponding non-linear wave equation (by perturbation) and obtaining the action of the dual theory on the boundary by inserting the solution into the action of massless scalars on Minkowski (Euclidean) space (the bulk). Furthermore the same method can be used to study (A)dS/CFT correspondence and the holographic description of Minkowski space-time in the case of higher spin fields [10].

5 Appendix

In this appendix we briefly review scalar field theory in $D = d + 1$ dimensional curved spacetime. The action for the scalar field ϕ is

$$S = \int d^D x \sqrt{|g|} \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (m^2 + \xi R) \phi^2 \right), \quad (31)$$

for which the equation of motion is

$$\left(\square + m^2 + \xi R \right) \phi = 0, \quad \square = |g|^{-1/2} \partial_\mu |g|^{1/2} g^{\mu\nu} \partial_\nu. \quad (32)$$

(With \hbar explicit, the mass m should be replaced by m/\hbar .) The case with $m = 0$ and $\xi = \frac{d-1}{4d}$ is referred to as conformal coupling [11].

Using Eq.(17) it is easy to show that the Ricci scalar R for dS_{d+1} space is $R = d(d+1)$ where we have set the dS radius $\ell = 1$. Therefore, the action for conformally coupled scalars in dS_{d+1} is

$$S = \int d^D x \sqrt{|g|} \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \left(\frac{d^2 - 1}{4} \right) \phi^2 \right), \quad (33)$$

Similar result can be obtained for AdS space using Eq.(1).

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